Even and Odd Functions

A function, f, is *even* (or *symmetric*) when

$$f(x) = f(-x).$$

A function, f, is *odd* (or *antisymmetric*) when

$$f(x) = -f(-x).$$

Even and Odd Functions (contd.)

Theorem 5.1 Any function can be written as a sum of even and odd functions.

$$
\begin{aligned}
f(t) &= \frac{1}{2}\left[f(t)+\underbrace{f(-t)-f(-t)}_{0}+f(t)\right] \\
&= \underbrace{\frac{1}{2}[f(t)+f(-t)]}_{f_e}+\underbrace{\frac{1}{2}[f(t)-f(-t)]}_{f_o}
\end{aligned}
$$

f_e is even because $f_e(t) = f_e(-t)$:

$$
\begin{aligned}
f_e(t) &= f(t)+f(-t) \\
&= f(-t)+f(t) \\
&= f_e(-t).
\end{aligned}
$$

f_o is odd because $f_o(t) = -f_o(-t)$:

$$
\begin{aligned}
f_o(t) &= f(t)-f(-t) \\
&= -[f(-t)-f(t)] \\
&= -f_o(-t).
\end{aligned}
$$

<u>Even and Odd Functions (contd.)</u>

Theorem 5.2 The integral of the product of odd and even functions is zero.

$$\int_{-\infty}^{\infty} f_e(x) f_o(x) dx =$$

$$\int_{-\infty}^{0} f_e(x) f_o(x) dx + \int_{0}^{\infty} f_e(x) f_o(x) dx.$$

Substituting $-x$ for x and $-dx$ for dx in the first term yields:

$$\int_{\infty}^{0} -f_e(-x) f_o(-x) dx + \int_{0}^{\infty} f_e(x) f_o(x) dx$$

$$= \int_{0}^{\infty} f_e(-x) f_o(-x) dx + \int_{0}^{\infty} f_e(x) f_o(x) dx$$

$$= \int_{0}^{\infty} \left[f_e(-x) f_o(-x) + f_e(x) f_o(x) \right] dx.$$

Substituting $f_e(-x)$ for $f_e(x)$ and $-f_o(-x)$ for $f_o(x)$ yields:

$$\int_{0}^{\infty} \underbrace{\left[f_e(-x) f_o(-x) - f_e(-x) f_o(-x) \right]}_{0} dx.$$

Fourier Transform Symmetry

The Fourier transform of $f(t)$ is defined to be:

$$F(s) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi st} dt.$$

This can be rewritten as follows:

$$F(s) = \int_{-\infty}^{\infty} f(t) \cos(2\pi st) dt - j \int_{-\infty}^{\infty} f(t) \sin(2\pi st) dt.$$

Substituting $f_e(t) + f_o(t)$ for $f(t)$ yields:

$$F(s) = \int_{-\infty}^{\infty} f_e(t) \cos(2\pi st) dt + \int_{-\infty}^{\infty} f_o(t) \cos(2\pi st) dt -$$

$$j \int_{-\infty}^{\infty} f_e(t) \sin(2\pi st) dt - j \int_{-\infty}^{\infty} f_o(t) \sin(2\pi st) dt.$$

Fourier Transform Symmetry (contd.)

However, the second and third terms are zero (Theorem 5.2):

$$F(s) = \int_{-\infty}^{\infty} f_e(t)\cos(2\pi st)dt - j\int_{-\infty}^{\infty} f_o(t)\sin(2\pi st)dt.$$

It follows that:

$$F(s) = F_e(s) + F_o(s).$$

Fourier Transform Symmetry (contd.)

Theorem 5.3 The Fourier transform of a real even function is real.

$$
\begin{aligned}
F(s) &= \int_{-\infty}^{\infty} f(t) e^{-j2\pi st} dt \\
&= \int_{-\infty}^{\infty} f(t) \left[\cos(2\pi st) + j\sin(2\pi st)\right] dt \\
&= \int_{-\infty}^{\infty} f(t)\cos(2\pi st) dt
\end{aligned}
$$

which is real.

Fourier Transform Symmetry (contd.)

Theorem 5.4 The Fourier transform of a real odd function is imaginary.

$$\begin{aligned}
F(s) &= \int_{-\infty}^{\infty} f(t) e^{-j2\pi st} dt \\
&= \int_{-\infty}^{\infty} f(t) \left[\cos(2\pi st) + j\sin(2\pi st)\right] dt \\
&= j\int_{-\infty}^{\infty} f(t) \sin(2\pi st) dt
\end{aligned}$$

which is imaginary.

Fourier Transform Symmetry (contd.)

Theorem 5.5 The Fourier transform of an even function is even.

$$F(s) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi st} dt$$

Subsituting $f(-t)$ for $f(t)$ yields:

$$F(s) = \int_{t=-\infty}^{t=\infty} f(-t) e^{-j2\pi st} dt.$$

Substituting u for $-t$ and $-du$ for dt yields:

$$\begin{aligned} &= \int_{u=\infty}^{u=-\infty} -f(u) e^{-j2\pi s(-u)} du \\ &= \int_{-\infty}^{\infty} f(u) e^{-j2\pi(-s)u} du \\ &= F(-s). \end{aligned}$$

Fourier Transform Symmetry (contd.)

Theorem 5.6 The Fourier transform of an odd function is odd.

$$F(s) = \int_{-\infty}^{\infty} f(t)e^{-j2\pi st}dt$$

Subsituting $-f(-t)$ for $f(t)$ yields:

$$F(s) = \int_{t=-\infty}^{t=\infty} -f(-t)e^{-j2\pi st}dt.$$

Substituting u for $-t$ and $-du$ for dt yields:

$$\begin{aligned}
&= \int_{u=\infty}^{u=-\infty} f(u)e^{-j2\pi s(-u)}du \\
&= \int_{-\infty}^{\infty} -f(u)e^{-j2\pi(-s)u}du \\
&= -F(-s).
\end{aligned}$$

Fourier Transform Symmetry (contd.)

- The Fourier transform of the even part (of a real function) is real (Theorem 5.3):

$$\mathcal{F}\{f_e\}(s) = F_e(s) = \mathrm{Re}(F_e(s)).$$

- The Fourier transform of the even part is even (Theorem 5.5):

$$\mathcal{F}\{f_e\}(s) = F_e(s) = F_e(-s).$$

- The Fourier transform of the odd part (of a real function) is imaginary (Theorem 5.4):

$$\mathcal{F}\{f_o\}(s) = F_o(s) = \mathrm{Im}(F_o(s)).$$

- The Fourier transform of the odd part is odd (Theorem 5.6):

$$\mathcal{F}\{f_o\}(s) = F_o(s) = -F_o(-s).$$

Fourier Transform Symmetry (contd.)

We can summarize all four symmetries possessed by the Fourier transform of a real function as follows:

$$\begin{aligned} F_e(s) + F_o(s) &= F_e(-s) - F_o(-s) \\ &= \left[F_e(-s) + F_o(-s)\right]^*. \end{aligned}$$

This is termed *Hermitian* (or *conjugate*) symmetry:

$$F(s) = F^*(-s).$$

where F is the Fourier transform of a real function f.